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Asynchronism interactions and their impact on consensus formation in online and offline complex networks

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Nowadays, with the development of information communications technology and Internet, more and more people receive information and exchange their opinions with others in online environment (e.g. Twitter, Facebook, Weibo, and WeChat). According to e-Marketer Report[30], by the end of 2016, more than 3.2 billion individuals worldwide will use the internet regularly, accounting for nearly 45% of the worlds population. In other words, the other half obtains information and exchanges their opinions via traditional way (e.g. face to face) regularly. Generally, the speed at which information spreads and opinions are exchanged and updated in online environment is much faster than in offline environment. In this paper, we study the asynchronism interaction between the

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online and offline environment in opinion dynamics, and we unfold that the asynchronization strongly impacts the consensus formation at complex networks: A high degree of the asynchronization makes it difficult for all agents to reach consensus in opinion dynamics. Furthermore, these effects are often further intensified as the number of online participating agents increases.

Keywords: opinion dynamics, consensus formation,asynchronism interactions,online and offline,complex networks

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1. Introduction

Consensus formation is an interesting problem which attracts much attention in natural and social sciences. Opinion dynamics is a complex process, and is generally used as a tool to investigate the consensus formation[1] . There exists different variants in opinion dynamics model, such as Ising model[2,3,4,5], Sznajd model[6,7,8], Voter model[9,10,11,12] , Majority rule model[13,14], , French model[15], DeGroot model[16,17,18] , Friedkin and Johnsen model[19]and bounded confidence model[20,21,22,23,24]. The outcomes of these previous studies on opinion dynamics avail us to understand the agreement or disagreement phenomena in human behaviors and to manage collective behaviors[25].

Network-related information plays a key role to analyze the consensus formation in opinion dynamics. With the development of digital technique, internet and such these information and communication technologies (ICTs), human beings not only have large databases on the topology of various real networks but also investigate these real networks by computing power. Some concepts occupy a prominent place in contemporary thinking about complex networks, e.g., Small world effect, Clustering and Degree distribution. What is more, Erdös-Rényi (ER) random networks[26,27], Watts-Strogatz (WS) small-world networks[28] and Barabási -Albert (BA) scale-free networks[29] have been widely used to describe the structure of many interpersonal relationships in real world.

According to eMarketer Report[30](2016), more than 3.2 billion individuals worldwide can receive information and exchange their opinions with others in online environment regularly, while the rest always obtain information and exchange their opinions in offline environment. The Internet technologies (e.g. Facebook, Myspace, WeChat, etc.) enable online agents to spread and share information in a more rapid way than the offline agents[31,32,33]. And thus they access information and exchange opinions asynchronously in an online and offline network. However, the existing studies regarding consensus formation in opinion dynamics generally assume that there exists a common clock shared by all agents to update their opinions according to the established rules[1,5,8,12,15,18,34]. The general theory of asynchronous systems has been supported in the specialized literatures[35,36,37,38,39]. However, to the best of our knowledge there is a shortage of research to date, which investigates the opinion dynamics and consensus formation problems under the presence of asynchronous interactions in network contexts. Therefore, this study is devoted to discussing the consensus formation at complex networks with asynchronism online and offline interactions, and to scientifically demonstrate the effect of asynchronism on consensus formation.

Through extensive computational simulations and analyses, in this paper, we unveil that the asynchronous interactions under online and offline interactions strongly impacts the consensus formation in opinion dynamics. Particularly, the low level of asynchronization strengthens the consensus formation but, on the contrary, a high level of asynchronization contributes to reduce the. The increase in size of the online subsystem often enhances these effects.

2. The asynchronous opinion dynamics model in online and offline interactions

2.1. *Model*

In this section, we propose the asynchronous opinion dynamics model in online and offline interactions based on the HK bounded confidence model[24]. Let $V = \{1, 2, \ldots, n\}$ be a set of agents, let $x_i^t \in R$ denotes the opinion of agent $i \in V$ at time t, and thus $X^t = (x_1^t, x_2^t, \ldots, x_n^t)^T$ be the opinion profile at time t. Let ε be the homogeneous bounded confidence of the agents, and $A = (a_{ij})_{n\times n}$ be the adjacency matrix of the network.

All agents $V = \{1, 2, \ldots, n\}$ are divided into two types: online agents and offline agents. Specifically, let V^{on} be the set of online agents and V^{off} be the set of offline agents, where $V^{on} \bigcup V^{off} = V$ and $V^{on} \bigcap V^{off} = \emptyset$,and let $r = \#V^{on}/n$ be the proportion of the online agents, where $\#$ is the cardinality of a finite set.

Let $T^{on} = \{0, 1, 2, \ldots\}$ and $T^{off} = \{0, T, 2T, \ldots\}$ be the sets of time instants, where $T \geq 1$ and $T \in N$. Clearly,$T^{off} \subset T^{on}$. When time $t \in T^{on}$ and $t \notin T^{off}$, only the online agents will update their opinions, and when $t \in T^{off}$, both the online and offline agents will update their opinions. And thus $T = 1$ represents synchronization and $T \geq 2$ asynchronization. Such that the higher T , the stronger the level of asynchronization between online and offline agents. In this study, we set $T \in [1, 100]$,and the results are similar when $T \geq 100$.

In the following, we propose the asynchronous opinion dynamics model with asynchronous interactions in the framework of bounded confidence as follows:

Case A: $t + 1 \in T^{on}$ and $t + 1 \notin T^{off}$, In this case, for any agent $i \in V^{on}$,he/she only communicate with other online agents at time t, and the confidence set $I^A(i, X^t)$ of the agent is determined as:

$$I^A(i, X^t) = \{j||x_i^t - x_j^t| \leq \varepsilon\,,\, a_{ij} = 1,\ j \in V^{on}\}. \tag{1}$$

Then,w_{ij}^t of agent i assigns to agent j at time t can be calculated as:

$$w_{ij}^t = \begin{cases} \frac{1}{\#I^A(i,X^t)+1}, & j \in I^A(i, X^t) \\ 0, & j \notin I^A(i, X^t) \end{cases}. \tag{2}$$

In addition, any agent $i \in A^{off}$,he/she does not communicate with other agents at time t and thus he/she will not update his/her opinion at time $t+1$. i.e., $x_i^{t+1} = x_i^t$. Overall, in this case, the updated opinion x_i^{t+1} is calculated as:

$$x_i^{t+1} = \begin{cases} \sum_{j=1}^{n} w_{ij}^t x_j^t, \; i \in V^{on} \\ \\ x_i^t, \qquad\qquad\quad i \in V^{off} \end{cases}. \tag{3}$$

where $w_{ii}^t = \frac{1}{\#I^A(i,X^t)+1}$.

Case B: $t+1 \in T^{on}$ and $t+1 \in T^{off}$, In this case, for any agent $i \in V$ can communicate with both the online and offline agents at time t, Thus, the confidence set $I^B(i, X^t)$ of the agent is determined as:

$$I^B(i, X^t) = \{j||x_i^t - x_j^t| \leq \varepsilon\,,\ a_{ij} = 1,\, j \in V\} \tag{4}$$

Then,w_{ij}^t of agent i assigns to agent j at time t is determined as:

$$w_{ij}^t = \begin{cases} \frac{1}{\#I^B(i,X^t)+1}, \; j \in I^B(i, X^t) \\ \\ 0, \qquad\qquad\quad j \notin I^B(i, X^t) \end{cases}, \tag{5}$$

In this case, the updated opinion x_i^{t+1} is calculated as:

$$x_i^{t+1} = \sum_{j=1}^{n} w_{ij}^t x_j^t. \tag{6}$$

where $w_{ii}^t = \frac{1}{\#I^B(i,X^t)+1}$.

Based on Cases A and B, for any agent i at time $t+1 \in T^{on}$, the updated opinion x_i^{t+1} is calculated as:

$$x_i^{t+1} = \begin{cases} \sum_{j=1}^{n} w_{ij}^t x_j^t, \; i \in V^{on},\, t+1 \notin T^{off} \\ \\ x_i^t, \qquad\qquad\quad i \in V^{off},\, t+1 \notin T^{off} \\ \\ \sum_{j=1}^{n} w_{ij}^t x_j^t, \; i \in V, \quad\;\; t+1 \in T^{off} \\ \end{cases}, \tag{7}$$

where w_{ij}^t is determined by Eqs.(1)-(2) in the case of $i \in V^{on}$ and $t+1 \notin T^{off}$, or is determined by Eqs.(4)-(5) in the case of $i \in V$ and $t+1 \in T^{off}$.

2.2. *Experimental simulation settings*

We randomly set $n \times r$ agents to be online agents and the rest of agents to be offline agents. The initial opinions of all agents are generated using a uniformly random distribution in [0,1]. The complex networks are connected. Otherwise, all agents cannot reach a consensus in the proposed model. We built complex networks with the same size and near-equal average degree, and thus we set $n = 200, p = 0.03$

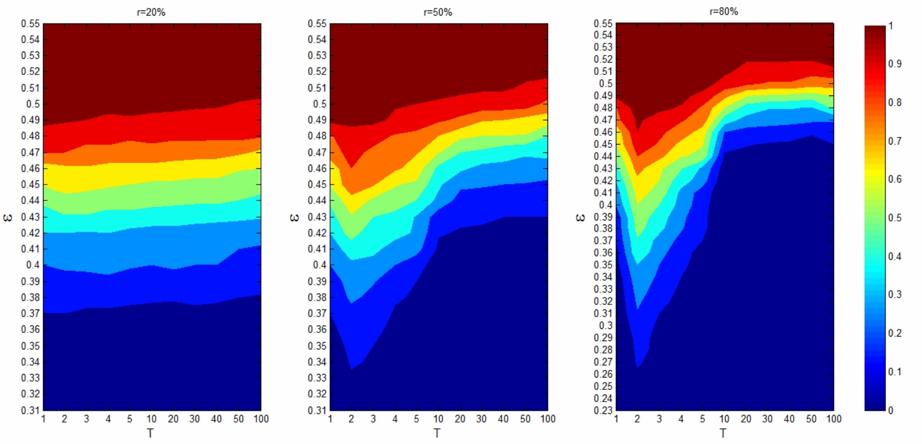

Fig. 1. Impact of T and ε on the consensus ratio in opinion dynamics on ER random networks, where $n = 200, p = 0.03$.

in ER random networks, i.e., we start with n agents, every pair of agents being connected with probability p. We set $n = 200, 2K = 6, p = 0.01$ in the WS small-world networks, i.e., we start with n agents in which every agent is connected to its first $2K$ neighbors (K on either side), then randomly rewire each edge of the lattice with probability p such that self-connections and duplicate edges are excluded.And we set$n = 200, m_0 = 6, m_1 = 3$ in the BA scale-free network, Starting with a small number m_0 of agents, at every time step we add a new agent with m_1 edges that link the new agent to m_0 different agents already present in the system.

Using the aforementioned models (1-7) proceeds the evolution of opinions, when the deviation between opinions of all the agents at two consecutive time instants is small enough, i.e.,$\parallel X_i^{t+1} - X_i^t \parallel \leq \delta$,the opinions of all agents reach the stable state, where $\parallel X \parallel_1 = \sum_{j=1}^{n} \mid x_i \mid$. Specially, we set $\delta = 10^{-5}$ in simulation. In additional, let x_i, x_j be the opinions of agents i, j when the opinions reach the stable state. We assign the agents i, j to the same cluster when $\mid x_i - x_j \mid \leq d$, and we set $d = 10^{-2}$ in simulation. The simulation results are averaged up to 1000 independent realizations for each set of parameters.

3. Results and discussion

3.1. *The consensus ratio*

The consensus ratio is the fraction of simulation samples with a single opinion cluster (i.e., all agents reach a consensus) at a stable state. As we know, increasing the confidence level yields more communications among agents in the Hegselmann-Krause model (i.e., HK model), and thus the high confidence level facilitates that all agents share the same opinion in the stable state. Figures 1-3 reveal the impact of T and ε on the consensus ratio at ER random networks, BA scale-free networks and

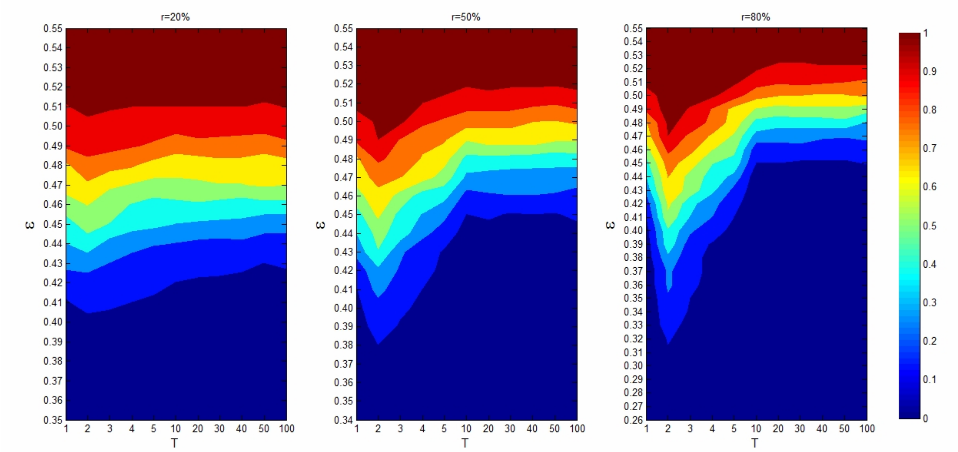

Fig. 2. Impact of T and ε on the consensus ratio in opinion dynamics on BA scale-free networks, where $n = 200, m_0 = 6, m_1 = 3$.

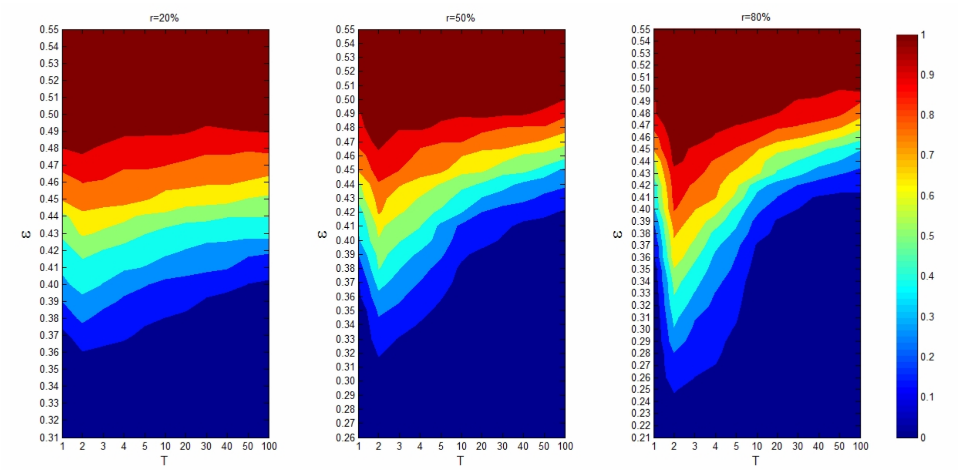

Fig. 3. Impact of T and ε on the consensus ratio in opinion dynamics on WS small-world networks, where $n = 200, 2K = 6, p = 0.01$.

WS small-world networks, respectively. As T increases, the consensus ratio starts increasing, then decreases and finally stabilizes under the certain confidence levels. Compared with the case of full synchronization (i.e.$T = 1$), the offline agents update their opinions more slowly than the online agents in the asynchronous case, and thus the difference between the opinions of online agents and offline agents starts increasing then stabilizes as T increases. In particular, when the level of asynchronization is low (i.e.$T = 2, 3$), the opinions of online agents cluster together quickly and the difference between the opinions of online agents and offline agents is small. Thus, the consensus will be easy to reach, and the consensus ratio increases under

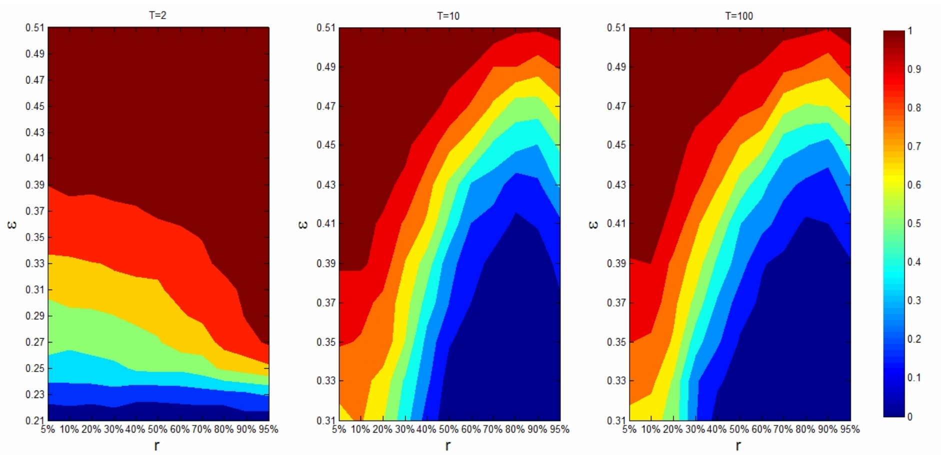

Fig. 4. Impact of r and ε on the consensus ratio in opinion dynamics on ER random networks, where $n = 200, p = 0.03$.

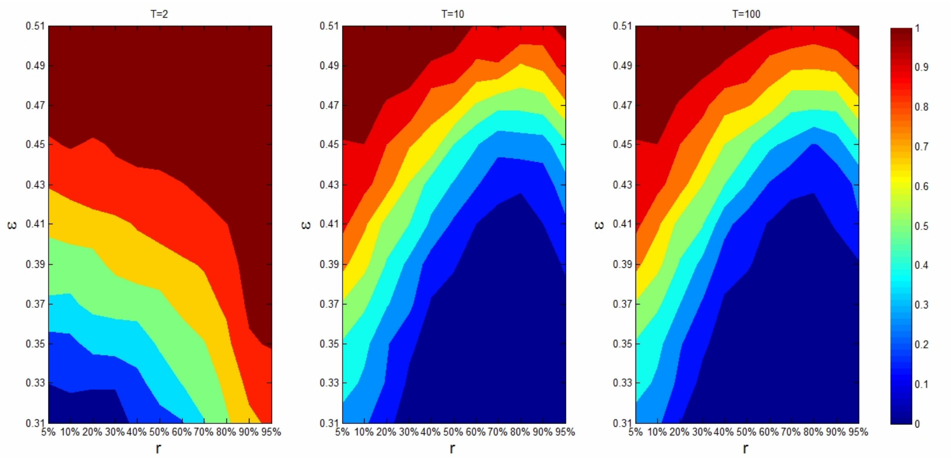

Fig. 5. Impact of Tr and ε on the consensus ratio in opinion dynamics on BA scale-free networks, where $n = 200, m_0 = 6, m_1 = 3$.

a certain confidence level. However, as the level of asynchronization increases, the opinions of online agents cluster together quickly and the difference between the opinions of online agents and offline agents becomes more significant, implying that consensus becomes more difficult to reach. Furthermore, if the level of asynchronization is very high (e.g.,$T = 100$), the difference between the opinions of online agents and offline agents does not get further increased as T becomes larger, which implies that the consensus ratio stabilizes under the different confidence levels.

Figures 4-6 further show the relationship between the size of the online subsystem and the consensus ratio under different degrees of asynchronization. The

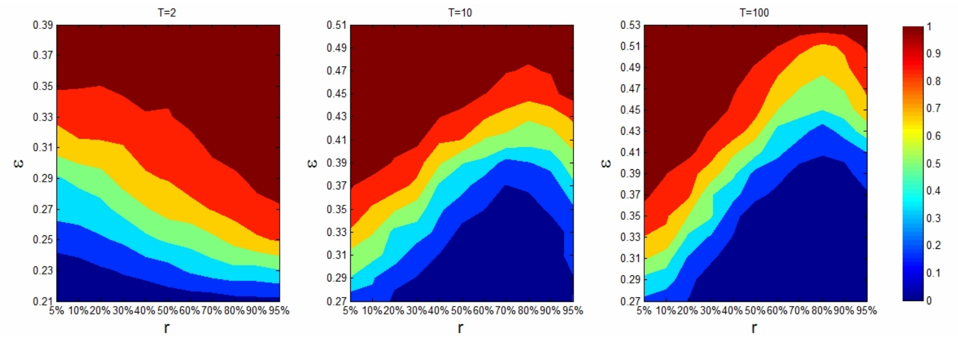

Fig. 6. Impact of r and ε on the consensus ratio in opinion dynamics on WS small-world networks, where $n = 200, 2K = 6, p = 0.01$.

differences between the consensus ratios under the low degree of asynchronization (e.g.,$T = 2$) and those under higher degrees of asynchronization (e.g.,$T = 10, 100$) are significant. A explanation for this phenomenon can be observed in the left panels of Figures 4-6 showing the relationship between the size of the online subsystem and the consensus ratio for the moderate confidence level: when $T = 2$, the larger the size of online agents is, the higher the consensus ratio is. However, the middle panel and right panel of Figures 4-6 further help understanding the impact of the size of online agents on the consensus ratio for the high degree of asynchronization(e.g.,$T = 10, 100$): when $r \leq 80\%$,the larger the size of online subsystem is, the lower the consensus ratio is. As commented earlier, the offline agents update their opinions more slowly than the online agents because of the asynchronization, the large size of online agent avails that the opinions of online agents cluster, and thus the increase in size of the online subsystem often enhances the above effects. However, if the proportion of online agents is much larger than the proportion of offline agents (e.g.$r > 80\%$), their opinions cluster together quickly and attract the offline agents easily, and thus the consensus ratio increases moderately with r further increasing under high degrees of asynchronization.

3.2. *The consensus threshold*

In particular, if the initial opinions of all agents are in [0,1] , there is a special ε_c so-called hereinafter consensus threshold, such that, for $\varepsilon \geq \varepsilon_c$, almost all agents can reach a consensus. If the average degree of the network stays finite when the order n of the network diverges, then $\varepsilon_c \approx 0.5$ in HK model[40]. According to eMarketer Report[30](2016), in the real world about half of the populations are online agents. Consequently, in the sequel we focus on the case of $r = 50\%$ as shown graphically in Figures 7-13. From the Figures 7-9, we find asynchronism interaction strongly impacts the consensus threshold compared with synchronization. On the one hand, the low level of asynchronization increases the consensus formation but the high level of asynchronization decreases for the reasons explained above. On the other hand,

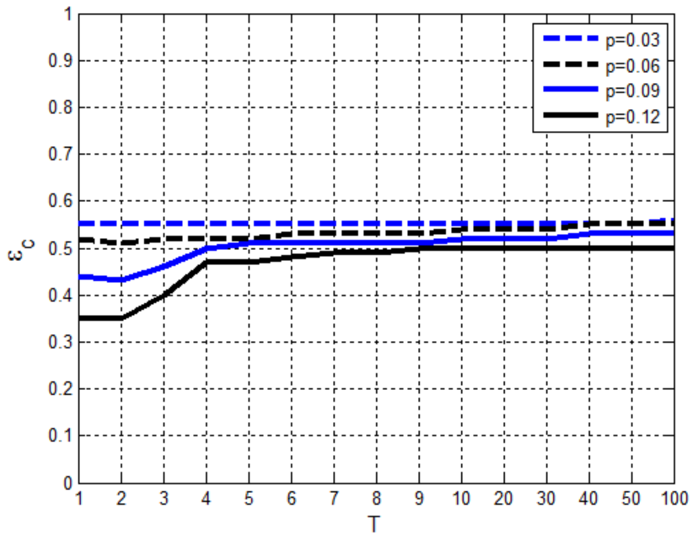

Fig. 7. Impact of T and p on ε_c in opinion dynamics on ER random networks, where $n = 200, r = 50\%$.

the consensus threshold decreases as the average degree of the networks increases. This occurs for the following reason: The increase in the average degree of the networks means that all agents communicate with others more easily, and thus the consensus threshold decreases.

In addition, it is known that the cluster coefficient of WS small-world networks is as follows:

$$C(p) = \tfrac{3(K-1)}{2(2K-1)}(1-p)^3.$$

In summary, the consensus threshold is always small at WS small-world networks with the higher cluster coefficient from the Figure 10 especially for $T > 6$. The higher cluster coefficient indicates all agents in the WS small-world networks tend to cluster together, and thus they only require small consensus threshold to reach a consensus.

Figures 11-13 further help us understand the impact of the proportion of online agents on the consensus threshold at different networks with the same size and the approximately equal average degree. These figures indicate that a larger proportion of online agents leads to a higher consensus threshold when T is large. The offline agents update their opinions more slowly than the online agents because of the asynchronization, and thus some offline agents will have less communications with other agents when T is large. Therefore, all agents need a higher confidence level to reach a consensus, and the large size of online agents clearly enhances these effects.

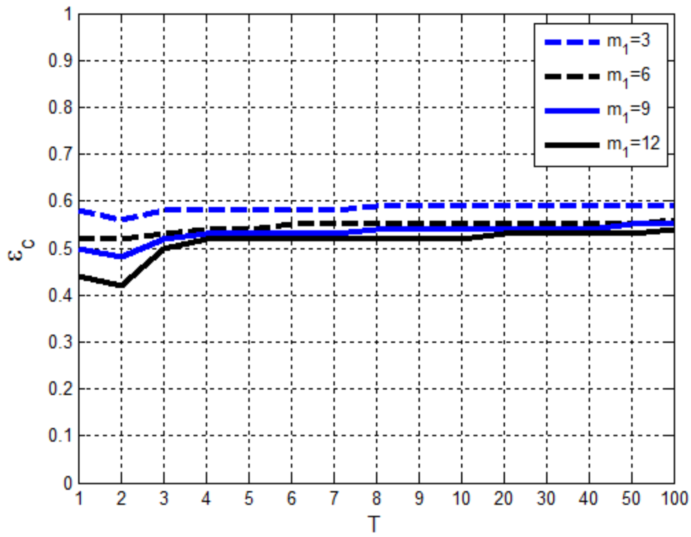

Fig. 8. Impact of T and m_1 on ε_c in opinion dynamics on BA scale-free networks, where $n = 200, m_0 = 18, r = 50\%$.

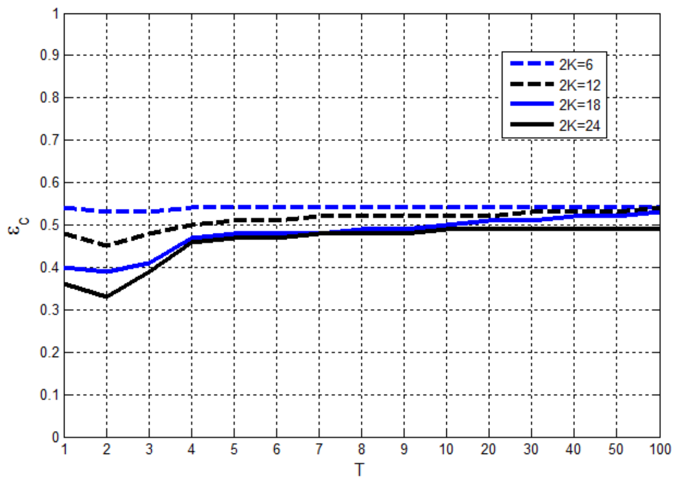

Fig. 9. Impact of T and $2K$ on ε_c in opinion dynamics on WS small-world networks, where $n = 200, p = 0.01, r = 50\%$.

However, when T is small (e.g.,$T = 2$), the results regarding the consensus threshold at WS small-world networks and ER random networks are similar, while the results at BA scale-free networks are different from them. The growth-based trend of the consensus formation with the low level of asynchronization at BA scale-free networks

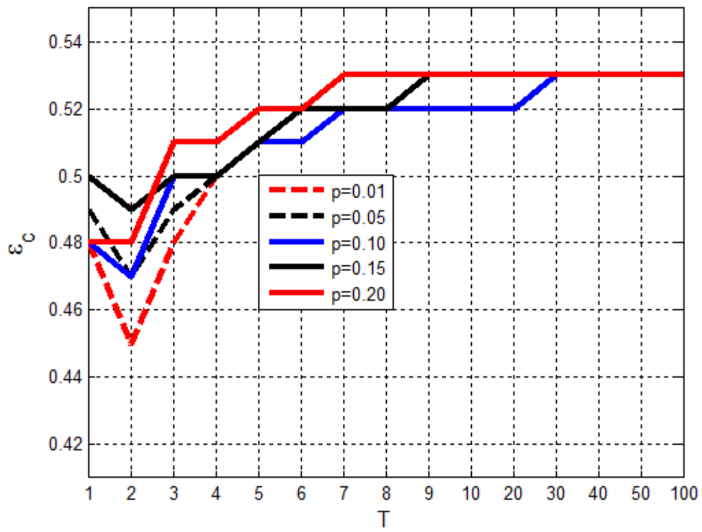

Fig. 10. Impact of T and p on ε_c in opinion dynamics on WS small-world networks, where $n = 200, 2K = 6, r = 50\%$.

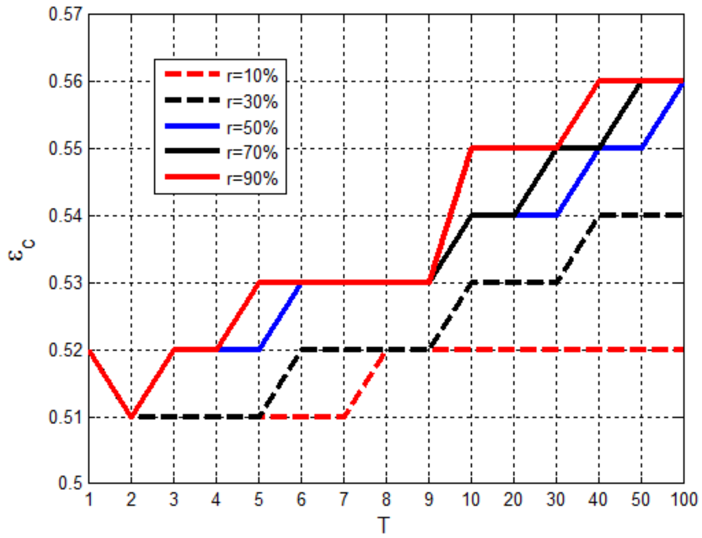

Fig. 11. Impact of T and r on ε_c in opinion dynamics on ER random networks, where $n = 200, p = 0.06$.

is not obvious. As we know, there are many agents who have few neighbors at BA scale-free networks. Naturally, it is highly likely that some offline agents have few

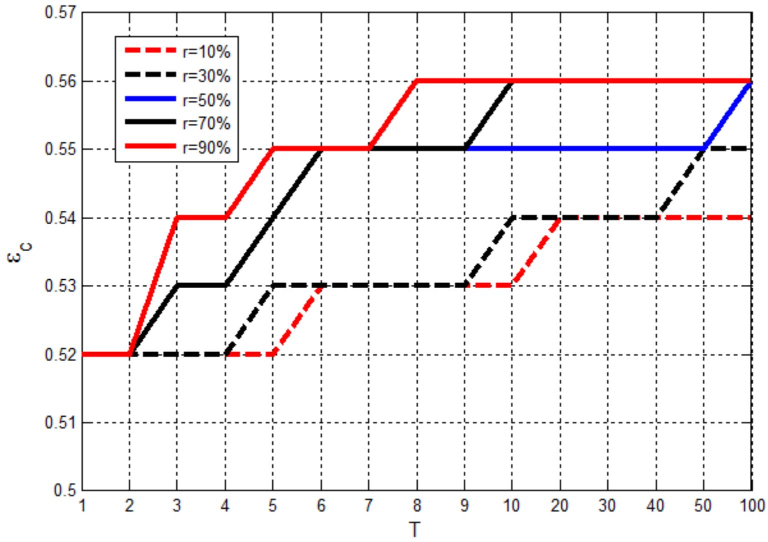

Fig. 12. Impact of T and r on ε_c in opinion dynamics on BA scale-free networks, where $n = 200, m_0 = 10, m_1 = 6$.

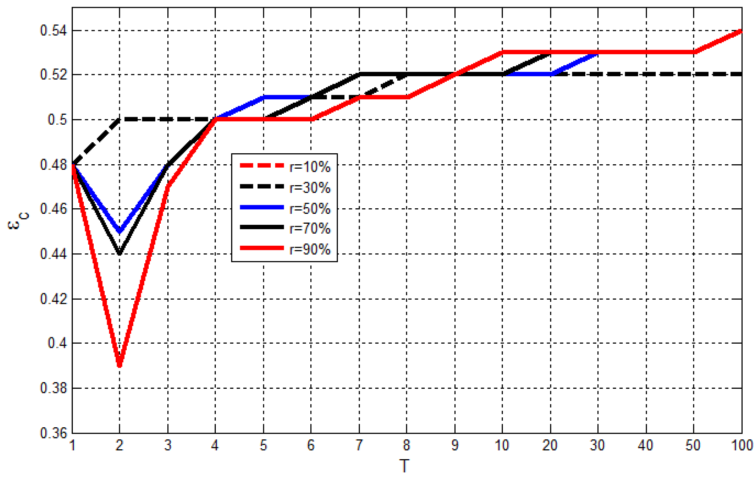

Fig. 13. Impact of T and r on ε_c in opinion dynamics on WS small-world networks, where $n = 200, 2K = 12, p = 0.01$.

neighbors, and these offline agents always do not communicate with other agents over the course of the time.

4. Conclusion

With the development of Information and Internet technology, the asynchronization is a very popular phenomenon in the evolution of real-world public opinions. In this study we investigate the opinion dynamics with asynchronization between online and offline agents at complex networks. Through extensive computational simulations and analyses, we unfold that the degree of asynchronization strongly impacts the consensus formation (namely, the consensus ratio and the consensus threshold), and that as the size of the online agent increases these effects are enhanced when the degree of the asynchronization is high. More important, we find that a low degree of the asynchronization (i.e.$T = 2$) contributes to the consensus formation, whereas a high degree of the asynchronization restrains the consensus formation, which can provide the decision support for the government to analyses the dynamics of public opinions such as, when the government attempts to analyses the dynamics of public opinions on introducing a policy, certain citizens may express opinions via the Internet, and the rest of citizens obtain information and exchange their opinions via the traditional approach. And thus governments should provide more supports to promote the interactions with some offline agents. Otherwise, it may occur that not all agents can reach a consensus on some issues because of the high degree of the asynchronization. Moreover, governments should make a full use of the low degree of the asynchronization if they want all or most people reach an agreement on some issues.

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